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NEAR FIELD STRESS ANALYSIS OF A SPOT WELD BETWEEN ELASTIC PLATES

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Abstract—The mechanics of a circular, spot weld between two identical elastic plates is studied. A method of solution similar to that used by the authors in a paper concerned with the influence of an elastic layer on the tangential compliance of bodies in contact is used. In the present analysis the relation between tangential load and deflection is lost due to the nature of the problem; hence, physical quantities are determined in terms of the total tangential load applied in the weld region.

1. INTRODUCTION

In this communication the mechanics of a spot weld between two identical elastic plates is studied. The weld is assumed to be circular and the bonding adhesive so thin that its effect is only to provide displacement continuity in the bonded region. Exterior to the bonded region and in the plane of the weld the plate surfaces are assumed to be stress-free. The outer plate surfaces (upper for top plate and lower for bottom plate) are also assumed to be stress-free.

Analogous problems have been solved for two dimensions. For example, the problem of a lap joint has been considered by Chang and Muki[1] and the cases of two and three layers bonded together have been considered by Keer[2] and Keer and Chantaramungkorn[3]. Descriptions of solutions for adhesive joints by means of engineering theory have been given in a survey article by Sneddon[4]. However, most of the previous literature in the area of adhesive mechanics deals with plane elastic problems. The present analysis will study the effect of the bonding of two identical plates by means of a circular weld and will therefore by three-dimensional in nature.

The case of tangential shear loading in the weld region will be studied and the method of analysis will be seen to be similar to that used by Goodman and Keer[5] in a paper dealing with the influence of an elastic layer on the tangential compliance of bodies in contact. One significant difference seen to occur in the present analysis is the inability to obtain a relation between load and deflection. A slightly different technique from that used in [5] is therefore used for the present analysis. This technique avoids the difficulty mentioned and allows the results to be given in terms of the applied tangential load directly. Such a problem was encountered in a paper by Keer[6] and the remarks made there will also be applicable here.

2. BOUNDARY CONDITIONS AND BASIC EQUATIONS

The present analysis deals with the stress distributions in a circular weld between two plates of identical materials and thicknesses when the plates are moved relative to each other in a tangential direction. The weld is of radius a and the plates are of thickness h. The two coordinate systems that will be used are cylindrical (r, θ, z) and rectangular (x, y, z) where the origin of the coordinate systems will be taken at the center of the weld with the z-axis perpendicular to the weld (see Fig. 1). Equal and opposite tractions in plane loading are applied to the upper and lower plates in the direction of the x-axis. From the symmetry of the loading conditions the normal stresses can be taken as zero in the plane of the weld. The boundary conditions for this problem are expressed in the following manner:

$$\tau_{zz} = u_y = 0,\ u_x = \Delta \qquad z = 0,\quad 0 \le r \le a \tag{1}$$

$$\tau_{zz} = \tau_{zx} = \tau_{zy} = 0 \qquad z = 0,\quad a < r < \infty \tag{2}$$

$$\tau_{zz} = \tau_{zx} = \tau_{zy} = 0 \qquad z = h,\quad 0 \le r < \infty \tag{3}$$

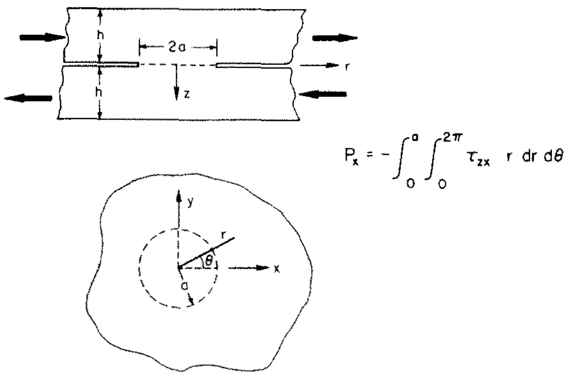

Fig. 1. Geometry and coordinate system.

where it is convenient to give the stresses and displacements in terms of a cylindrical coordinate system as

$$\tau_{zx} = \tau_{zr}\cos\theta - \tau_{z\theta}\sin\theta \tag{4a}$$

$$\tau_{zy} = \tau_{zr}\sin\theta + \tau_{z\theta}\cos\theta \tag{4b}$$

$$u_x = u_r\cos\theta - u_\theta\sin\theta \tag{5a}$$

$$u_y = u_y\sin\theta + u_\theta\cos\theta \tag{5b}$$

The stresses and displacements in cylindrical coordinates are further written in terms of their Fourier components as

$$\tau_{zr} = \sum_{n=0}^{\infty} \tau_{zr}^{n}\cos n\theta \tag{6a}$$

$$\tau_{z\theta} = \sum_{n=1}^{\infty} \tau_{z\theta}^{n}\sin n\theta \tag{6b}$$

$$\tau_{zz} = \sum_{n=0}^{\infty} \tau_{zz}^{n}\cos n\theta \tag{6c}$$

$$u_r = \sum_{n=0}^{\infty} u_r^{n}\cos u\theta \tag{7a}$$

$$u_\theta = \sum_{n=1}^{\infty} u_\theta^{n}\sin n\theta \tag{7b}$$

$$u_z = \sum_{n=0}^{\infty} u_z^{n}\cos n\theta \tag{7c}$$

A displacement solution to the field equations of the classical theory of elasticity can be easily established in terms of integral transforms that satisfies the stress-free condition on $z = h$, eqn (3), as well as the condition that the normal stress is zero on $z = 0$. Such a solution is given in terms of the Fourier coefficients of the displacements in polar coordinates as

$$\begin{aligned} 2\mu(u_r^n + u_\theta^n) = \frac{1}{2}\int_0^\infty \{&(S_n - T_n)p^{-1}[(\beta^2 - 2(1-\nu)\,\mathrm{sh}^2\beta - \xi z g)\,\mathrm{sh}(\xi z) \\ &-(\xi z\,\mathrm{sh}^2\beta + 2(1-\nu)g)\,\mathrm{ch}(\xi z)] - 2(S_n + T_n) \\ &\times[\mathrm{ctnh}\,\beta\,\mathrm{ch}(\xi z) - \mathrm{sh}(\xi z)]\}J_{n+1}(\xi r)\,d\xi \end{aligned} \tag{8a}$$

$$2\mu(u_r^n - u_\theta^n) = \frac{1}{2}\int_0^\infty \{(S_n - T_n)p^{-1}[\xi z g + 2(1-\nu)\mathrm{sh}^2\beta - \beta^2)\,\mathrm{sh}(\xi z) + (\xi z\,\mathrm{sh}^2\beta - 2(1-\nu)g)\,\mathrm{ch}(\xi z)]$$
$$- 2(S_n + T_n)\cdot[\mathrm{ctnh}\,\beta\,\mathrm{ch}(\xi z) - \mathrm{sh}(\xi z)]\}J_{n-1}(\xi r)\,\mathrm{d}\xi \tag{8b}$$

$$2\mu u_z^n = \frac{1}{2}\int_0^\infty (S_n - T_n)p^{-1}[\xi z g - ((1-2\nu)\mathrm{sh}^2\beta - \beta^2)\,\mathrm{ch}(\xi z)$$
$$+ (\xi z\,\mathrm{sh}^2\beta - (1-2\nu)g\,\mathrm{sh}(\xi z)]J_n(\xi r)\,\mathrm{d}\xi \tag{8c}$$

where

$$p = \beta^2 - \mathrm{sh}^2\beta \tag{9a}$$

$$g = \beta - \mathrm{sh}\,\beta\,\mathrm{ch}\,\beta \tag{9b}$$

and

$$\beta = \xi h. \tag{9c}$$

The stresses are easily computed from Hooke's law and are given by

$$\tau_{zr}^n + \tau_{z\theta}^n = \frac{1}{2}\int_0^\infty \{(S_n - T_n)p^{-1}[(p - \xi z g)\,\mathrm{ch}(\xi z) - (\xi z\,\mathrm{sh}^2\beta + g)\,\mathrm{sh}(\xi z)]$$
$$- (S_n + T_n)[\mathrm{ctnh}\,\beta\,\mathrm{sh}(\xi z) - \mathrm{ch}(\xi z)\}\xi J_{n+1}(\xi r)\,\mathrm{d}\xi \tag{10a}$$

$$\tau_{zr}^n - \tau_{z\theta}^n = \frac{1}{2}\int_0^\infty \{(S_n - T_n)p^{-1}[(\xi z g - p)\mathrm{ch}(\xi z) + (\xi z\,\mathrm{sh}^2\beta + g)\,\mathrm{sh}(\xi z)]$$
$$- (S_n + T_n)[\mathrm{ctnh}\,\beta\,\mathrm{sh}(\xi z) - \mathrm{ch}(\xi z)]\}\xi J_{n-1}(\xi r)\,\mathrm{d}\xi \tag{10b}$$

$$\tau_{zz}^n = \frac{1}{2}\int_0^\infty (S_n - T_n)p^{-1}[(\xi z g - \beta^2)\,\mathrm{sh}(\xi z) + \xi z\,\mathrm{sh}^2\beta\,\mathrm{ch}\,\xi z]\xi J_n(\xi r)\,\mathrm{d}\xi \tag{10c}$$

where in the above equations $\mathrm{sh}(x)$, $\mathrm{ch}(x)$ are hyperbolic sine and cosine.

The relevant surface stresses and displacements on $z = 0$ are given in the following form:

$$\tau_{zr}^n + \tau_{z\theta}^n = \int_0^\infty S_n \xi J_{n+1}(\xi r)\,\mathrm{d}\xi \tag{11a}$$

$$\tau_{zr}^n - \tau_{z\theta}^n = \int_0^\infty T_n \xi J_{n-1}(\xi r)\,\mathrm{d}\xi \tag{11b}$$

$$\tau_{zz}^n = 0. \tag{11c}$$

$$2\mu(u_r^n + u_\theta^n) = -\int_0^\infty [\nu T_n + (2-\nu)S_n]J_{n+1}(\xi r)\,\mathrm{d}\xi + \frac{1}{2}\int_0^\infty [M - N)T_n - (M+N)S_n]J_{n+1}(\xi r)\,\mathrm{d}\xi \tag{12a}$$

$$2\mu(u_r^n - u_\theta^n) = -\int_0^\infty [(2-\nu)T_n - \nu S_n]J_{n-1}(\xi r)\,\mathrm{d}\xi + \frac{1}{2}\int_0^\infty [-(M+N)T_n + (M-N)S_n]J_{n-1}(\xi r)\,\mathrm{d}\xi \tag{12b}$$

where

$$M = 2(1-\nu)(g/p - 1) \tag{13a}$$

$$N = 2(\mathrm{ctnh}\,\beta - 1). \tag{13b}$$

Here, it is noted that

$$N, M = 0(e^{-2\beta}) \quad \text{as} \quad \beta \to \infty \tag{14a}$$

$$N, M = 0(\beta^{-1}) \quad \text{as} \quad \beta \to 0. \tag{14b}$$

Equations (11) and (12), when used with the boundary conditions, (1) and (2), lead to a coupled pair of dual integral equations which may be solved by a modification of a technique used by Westmann[7]. Thus set

$$S_1(\xi) = \xi^{1/2} \int_0^a t^{1/2} \phi_3(t) J_{3/2}(\xi t)\, dt \tag{15a}$$

$$T_1(\xi) = \xi^{1/2} \int_0^a t^{1/2} \phi_1(t) J_{-1/2}(\xi t)\, dt - \frac{\nu}{2-\nu} \xi^{1/2} \int_0^a t^{1/2} \phi_3(t) J_{3/2}(\xi t)\, dt. \tag{15b}$$

Equations (15) lead, after an integration by parts and the use of certain identities (see e.g. [5]), to the following expressions for the surface stresses within the weld region,

$$\tau^1_{zr} + \tau^1_{z\theta} = \sqrt{\left(\frac{2}{\pi}\right)}\, r^2 \left\{ \frac{a^{-2}\phi_3(a)}{(a^2 - r^2)^{1/2}} - \int_r^a \frac{[t^{-2}\phi_3(t)]'}{(t^2 - r^2)^{1/2}}\, dt \right\}, \quad 0 \le r \le a \tag{16a}$$

$$\tau^1_{zr} - \tau^1_{z\theta} = \sqrt{\left(\frac{2}{\pi}\right)} \left\{ \frac{1}{(a^2 - r^2)^{1/2}} \left[\phi_1(a) + \frac{\nu}{2-\nu} \phi_3(a) \right] \right.$$

$$\left. \int_r^a \left[\phi_1'(t) + \frac{\nu}{2-\nu} t^{-1}(t\phi_3(t))' \right] \frac{dt}{(t^2 - r^2)^{1/2}}, \quad 0 \le r \le a \right. \tag{16b}$$

Exterior to this region, the stresses are automatically zero. An elementary integration of the surface stresses establishes that the applied load given in terms of the auxiliary functions becomes

$$P_x = -\sqrt{(2\pi)} \int_0^a \phi_1(t)\, dt. \tag{17}$$

If one uses eqns (15) together with the boundary conditions for the displacements, eqns (1), one is immediately led by use of Westmann's solution to a coupled pair of integral equations for the auxiliary functions, ϕ_1 and ϕ_3. However, the kernels of these integral equations have integrands which diverage at the lower limit in a manner similar to that in Ref. [6]. It is therefore necessary to integrate by parts the integral transforms S_1 and T_1 as follows:

$$S_1(\xi) = -\sqrt{\left(\frac{2}{\pi}\right)}\, \xi^{-1} \left[\phi_3(a) \sin(\xi a) - \int_0^a \psi_3(t) \sin(\xi t)\, dt \right] \tag{18a}$$

$$T_1(\xi) = \sqrt{\left(\frac{2}{\pi}\right)}\, \xi^{-1} \left[\phi_1(a) \sin(\xi a) - \int_0^a \psi_1(t) \sin(\xi t)\, dt \right]$$

$$- \frac{\nu}{2-\nu} \sqrt{\left(\frac{2}{\pi}\right)}\, \xi^{-1} \left[\phi_3(a) \sin(\xi a) - \int_0^a \psi_3(t) \sin(\xi t)\, dt \right] \tag{18b}$$

where

$$\psi_1(t) = \phi_1'(t) \tag{19a}$$

$$\psi_3(t) = t^{-1}[t\phi_3(t)]'. \tag{19b}$$

Using the forms of the integral transforms as defined by eqns (18) with the notation of eqns (19) together with the displacement boundary conditions (1) and the displacements (12), one applies the method of Westmann [7] with some minor modifications to obtain the following coupled pair of Fredholm integral equations of the second kind:

$$\pi(2-\nu)\psi_1(s)-\phi_1(a)\int_0^\infty (M+N)\sin(\xi s)\sin(\xi a)\,d\xi$$
$$+\int_0^a \psi_1(t)\int_0^\infty (M+N)\sin(\xi s)\sin(\xi t)\,d\xi\,dt$$
$$-\frac{2}{2-\nu}\left\{\phi_3(a)\int_0^\infty [M-(1-\nu)N]\sin(\xi a)\sin(\xi s)\,d\xi\right.$$
$$\left.-\int_0^a \psi_3(t)\int_0^\infty [M-(1-\nu)N]\sin(\xi t)\sin(\xi s)\,d\xi\,dt\right\}=0,\quad 0\le s\le a \tag{20a}$$

$$\pi\nu\psi_1(s)+\phi_1(a)\int_0^\infty (M-N)\sin(\xi a)\sin(\xi s)\,d\xi-\int_0^a \psi_1(t)\int_0^\infty (M-N)\sin(\xi t)\sin(\xi s)\,d\xi\,dt$$
$$-\frac{4\pi(1-\nu)}{2-\nu}\psi_3(s)+\frac{2}{2-\nu}\left\{\phi_3(a)\int_0^\infty [M+(1-\nu)N]\sin(\xi a)\sin(\xi s)\,d\xi\right.$$
$$\left.-\int_0^a \psi_3(t)\int_0^\infty [M+(1-\nu)N]\sin(\xi t)\sin(\xi s)\,d\xi\,dt\right. = 0,\quad 0\le s\le a. \tag{20b'}$$

Equation (20b′) is not written in its most desirable form since the two unknown functions appear as a linear combination. The function $\psi_1(s)$ can be easily eliminated from (20b′) by use of (20a) to obtain

$$-4\pi(1-\nu)\psi_3(s)+\phi_1(a)\int_0^\infty [(3-\nu)M-(1-\nu)N]\sin(\xi a)\sin(\xi s)\,d\xi$$
$$-\int_0^a \psi_1(t)\int_0^\infty [(3-\nu)M-(1-\nu)N]\sin(\xi t)\sin(\xi s)\,d\xi\,dt$$
$$+\frac{2}{2-\nu}\phi_3(a)\int_0^\infty [(3-\nu)M+(1-\nu)^2N]\sin(\xi a)\sin(\xi s)\,d\xi$$
$$-\frac{2}{2-\nu}\int_0^a \psi_3(t)\int_0^\infty [(3-\nu)M+(1-\nu)^2N]\sin(\xi t)\sin(\xi s)\,d\xi\,dt=0,\quad 0\le s\le a. \tag{20b}$$

It is noted that eqns (20) no longer depend upon the displacement prescribed by eqn (1). Instead the results depend explicitly upon the quantities $\phi_1(a)$ and $\phi_3(a)$, which can be considered as known constants. They are determined by the following conditions:

$$P_x=-\sqrt{(2\pi)}\int_0^a \phi_1(t)\,dt=-\sqrt{(2\pi)}\left[a\phi_1(a)-\int_0^a t\psi_1(t)\,dt\right] \tag{21a}$$

$$a\phi_3(a)=\int_0^a t\psi_3(t)\,dt \tag{21b}$$

where

$$\phi_1(t)=\phi_1(a)-\int_t^a \psi_1(\tau)\,d\tau \tag{22a}$$

$$t\phi_3(t)=a\phi_3(a)-\int_t^a \tau\psi_3(\tau)\,d\tau. \tag{22b}$$

Since $\psi_1(t)$ and $\psi_3(t)$ are proportional to $\phi_1(a)$ and $\phi_3(a)$, eqns (21) provide the means for the determination of all results in terms of the applied load P_x. The significant quantity of interest for this problem is the stress intensity factor. Using the definition as given in the text by Sneddon and

Lowengrub[8], the stress intensity factors will be defined as follows:

$$K_1 = \lim_{r\to a} \sqrt{\left(\frac{2(a-r)}{a}\right)} \frac{(\tau^1_{zr} - \tau^1_{z\theta})}{(P_x/\pi a^2)} \tag{23a}$$

$$= (2\pi)^{1/2}(a/P_x)\left[\phi_1(a) + \frac{\nu}{2-\nu}\phi_3(a)\right] \tag{23b}$$

$$K_2 = \lim_{r\to a} \sqrt{\left(\frac{2(a-r)}{a}\right)} \frac{(\tau^1_{zr} + \tau^1_{z\theta})}{(P_x/\pi a^2)} \tag{24a}$$

$$= (2\pi)^{1/2}(a/P_x)\phi_3(a). \tag{24b}$$

The functions ψ_1, ψ_3 are computed from eqns (20a,b) in terms of the constants $\phi_1(a)$ and $\phi_3(a)$ by means of the iterative technique developed in [5]; then the constants $\phi_1(a)$ and $\phi_3(a)$ are determined by means of eqns (21a, b) and (22a, b) and, finally, the stress intensity factors K_1 and K_2 follow from (23b) and (24b). The values of the stress intensity factors are given in Table 1 below. The values for K_1 and K_2 represent, respectively, the symmetric and asymmetric portions of the stress intensity factors in the x and y directions.

It is useful to also present the results in terms of the conventional mode II and mode III stress intensity factors. These are given as

$$K_{II} = \lim_{r\to a}\{\sqrt{(2\pi(a-r))}\tau^1_{zr}\},\ K_{III} = \lim_{r\to a}\{-\sqrt{(2\pi(a-r))}\tau^1_{z\theta}\} \tag{25a,25b}$$

which in terms of K_1 and K_2 above may be written as

$$K_{II} = K^*_{II}P_x\pi^{-1/2}a^{-3/2}\cos\theta = \frac{1}{2}(K_1 + K_2)\cos\theta \tag{26a}$$

$$K_{III} = K^*_{III}P_x\pi^{-1/2}a^{-3/2}\cos\theta = \frac{1}{2}(K_1 - K_2)\sin\theta. \tag{26b}$$

Numerical values of K^*_{II} and K^*_{III} are given in Table 2.

3. DISCUSSION

The results presented in this paper are representative of the shear load of a spot weld with only near field considerations taken into account. Some mention of the far field loading conditions is required. If the load P_x were applied at $z = h/2$, then all loading would be in-plane. The far field load for this case is identical to the plane stress solution for a concentrated load and would diminish as $0(1/R)$ as $R = (x^2 + y^2)^{1/2} \to \infty$. The displacement is $0(\log R)$ and therefore becomes unbounded at

Table 1. Values of the stress concentration factors K_1 and K_2

a/h	υ = 0		υ = 0.3		υ = 0.5	
	K_1	K_2	K_1	K_2	K_1	K_2
0	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000
0.2	1.0033	0.0039	1.0036	0.0041	1.0039	0.0043
0.4	1.0235	0.0262	1.0255	0.0276	1.0277	0.0288
0.6	1.0715	0.0696	1.0754	0.0736	1.0802	0.0770
0.8	1.1592	0.1232	1.1608	0.1322	1.1654	0.1388
1.0	1.3148	0.1749	1.2995	0.1932	1.2944	0.2041
1.2	1.6206	0.2072	1.5421	0.2468	1.4992	0.2675
1.5	3.5287		2.6110	0.2832	2.2194	0.3502

Table 2. Values of the stress intensity factors K^*_{II} and K^*_{III}. ($K_{II} = K^*_{II}P_x\pi^{-1/2}a^{-3/2}\cos\theta$, $K_{III} = K^*_{III}P_x\pi^{-1/2}a^{-3/2}\sin\theta$)

a/h	$\nu = 0$		$\nu = 0.3$		$\nu = 0.5$	
	K^*_{II}	K^*_{III}	K^*_{II}	K^*_{III}	K^*_{II}	K^*_{III}
0	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
0.2	0.5036	0.4997	0.5039	0.4998	0.5041	0.4998
0.4	0.5249	0.4987	0.5266	0.4990	0.5283	0.4995
0.6	0.5706	0.5010	0.5745	0.5009	0.5786	0.5016
0.8	0.6412	0.5180	0.6465	0.5143	0.6521	0.5133
1.0	0.7449	0.5200	0.7464	0.5532	0.7493	0.5452
1.2	0.9139	0.7067	0.8945	0.6477	0.8834	0.6159
1.5			1.4471	1.1639	1.2848	0.9346

an infinitely large distance from the weld. This feature accounts for the lack of a relation between load and deflection mentioned earlier.

Since the load is applied on the surface of the layer, $z = 0$, the loading is also equipollent to a concentrated moment applied at the center of the layer. At a large distance from the origin the solution should be equivalent to the plate bending solution for a concentrated moment. The deflection for such a loading is given as

$$u_z = w = 0(R \log R \cos\theta) \quad \text{as} \quad R \to \infty.$$

Thus, the slope will be infinite at an infinitely large distance from the origin, and there will be no relationship between moment and rotation. Equation (8c) differentiated appropriately will have a divergent infinite integral as $\xi \to 0$, which indicates this behavior.

If one desired to produce a solution which gives a relation between load and deflection and between slope and moment, one would have to impose conditions on a layer of finite extent, say having a radius R. However, the present analysis should be adequate for demonstrating near field stress distributions for a spot weld.

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REFERENCES

1. D. J. Chang and R. Muki, Stress distribution in a lap joint under tension shear. *Int. J. Solids Structures* **10**, 503–517 (1974).
2. L. M. Keer, Stress analysis for bonded layers. *J. Appl. Mech.* **41**, 679–683 (1974).
3. L. M. Keer and K. Chantaramungkorn, Stress analysis for a double lap joint. *J. Appl. Mech.* **42**, 683–687 (1975).
4. I. N. Sneddon, *Adhesion* (Edited by D. D. Eley). Oxford University Press (1961).
5. L. E. Goodman and L. M. Keer, Influence of an elastic layer on the tangential compliance of bodies in contact. *Proc. IUTAM Synposium on Contact Problems*, Twente, The Netherlands (1974), pp. 127–151. Delft University Press (1975).
6. L. M. Keer, Two potential theory problems for a semi-infinite cylinder. *Int. J. Engng. Sci.* **14**, 1–6 (1976).
7. R. A. Westmann, Simultaneous pairs of dual integral equations. *SIAM Rev.* **7**(3), 19–24 (1961).
8. I. N. Sneddon and M. Lowengrub, *Crack Problems in the Classical Theory of Elasticity*. Wiley, New York (1969).

APPENDIX

Although the calculation was not performed in this paper, the following method can be used to enhance convergence of the infinite integrals in eqns (20). The integrals will be of the form

$$I(u) = \int_0^\infty M(\beta)\cos(\beta u)\,d\beta \tag{A1}$$

$$J(u) = \int_0^\infty N(\beta)\cos(\beta u)\,d\beta. \tag{A2}$$

The integral $I(u)$ can be written as follows:

$$I(u) = \int_0^\infty (M - M_\infty)\cos(\beta u)\,d\beta + \int_0^\infty M_\infty \cos(\beta u)\,d\beta \tag{A3}$$

where

$$M_\infty = -4\left(\beta - \beta^2 - \frac{1}{2}\right) e^{-2\beta} \tag{A4}$$

and hence

$$\int_0^\infty M_\infty \cos(\beta u)\, d\beta = 8(1-\nu)\left\{\frac{-1}{4+u^2} + \frac{4-u^2}{[4+u^2]^2} - \frac{4[4-3u^2]}{[4+u^2]^3}\right\}. \tag{A5}$$

Similarly,

$$J(u) = \int_0^\infty (N - N_\infty)\cos(\beta u)\, d\beta + 4/(4+u^2) \tag{A6}$$

where

$$N_\infty = 2\, e^{-2\beta}. \tag{A7}$$